CBM
R
8414
1994
NR.35

entER
for
omic Research

Discussion paper

CIN01585

TILBURG UNIVERSITY

Center
for
Economic Research

8414
1994
35

44

No. 9435

BIRD'S TREE ALLOCATIONS REVISITED

by Vincent Feltkamp,
Stef Tijs and Shigeo Muto

May 1994

ISSN 0924-7815

K.U.B.
BIBLIOTHEEK
TILBURG

Bird's tree allocations revisited

by
Vincent Feltkamp[1] Stef Tijs
CentER and Econometrics Department, Tilburg University,
P.O. Box 90153, 5000 LE Tilburg, The Netherlands
Shigeo Muto[2]
Faculty of Economics, Tohoku University, Kawauchi,
Aoba-ku, Sendai 980, Japan.

April 25, 1994

Abstract

Minimum cost spanning tree (mcst) construction and cost allocation problems have been studied extensively in the literature, though usually not together. Bird (1976) proposes an allocation rule of which Granot and Huberman (1981) prove that it lies in the core of the associated mcst game. We show that the problems of finding an mcst and allocating its cost can be integrated. Furthermore, we provide an axiomatic characterization of the set of all Bird tree allocations using consistency and converse consistency, and give a strategic form game of which the set of Nash equilibria contains Bird's tree allocations.

1 Introduction

Consider a group of villages, each of which needs to be connected directly or via other villages to a source. Such a connection needs costly links. Each village could connect itself directly to the source, but by cooperating costs might be reduced. This cost minimization problem is an old problem in Operations Research, and Borůvka (1926) came up with algorithms to construct a tree connecting every village to the source with minimal total cost. Later, Kruskal (1956), Prim (1957) and Dijkstra (1959) found similar algorithms. A historic overview of this minimization problem can be found in Graham and Hell (1985).

[1]This author is sponsored by the Foundation for the Promotion of Research in Economic Sciences, which is part of the Dutch Organization for Scientific Research (NWO). Email: v.feltkamp@kub.nl.

[2]This author wishes to acknowledge the Canon Foundation in Europe Visiting Research Fellowship which made his visit to CentER possible.

However, finding a minimal cost spanning tree (mcst) is only part of the problem : if the cost of this tree has to be borne by the villages, then a cost allocation problem has to be addressed as well. Claus and Kleitman (1973) introduced this cost allocation problem, whereupon Bird (1976) treated this problem with game-theoretic methods and for each minimum cost spanning tree proposed a cost allocation associated to it. We call the allocations yielded by this rule *Bird's tree allocations.* As there can be more than one mcst for a given problem, Bird's rule can yield more than one allocation. However, generically, there is only one mcst and then this rule yields a unique allocation.

Granot and Huberman (1981) proved that Bird's tree allocations are extremal points of the core of the associated minimum cost spanning tree game. This game is defined as follows : the players are the villages and the worth of a coalition is the minimal cost of connecting this coalition to the source via links between members of this coalition. Not being satisfied with only one extremal point of the core, Granot and Huberman then provide the weak and strong demand operations, which yield more core elements when applied to Bird's tree allocations. Aarts (1992) found other extreme points of the core in case the mcst problem has an mcst that is a *chain*, i.e. a tree with only two leaves. Kuipers (1993) investigated the core of information graph games. These are games arising from mcst situations in which the costs of links are either one or zero.

The reason for looking at other core allocations than those yielded by Bird's rule, is that although core elements are stable against defection by subcoalitions, an extremal point of the core discriminates against some players. For example, Bird's tree allocations discriminate against the players closest to the root. Granot and Huberman's demand operations remedy this problem by allowing players to demand contributions from players that 'need' them.

In this paper we provide two arguments for the defense of Bird's tree allocations : an axiomatic characterization of the set of Bird's tree allocations, and a non-cooperative game, in which Bird's tree allocations correspond to Nash equilibria.

The outline of this paper is as follows.

Section 2 presents minimum cost spanning tree problems and Bird's rule. Instead of solving the Operations Research and cost allocation problems consecutively, they can be more closely integrated : the cost of a link in an mcst can be allocated at the same moment the link is constructed in the process of forming the tree. Section 3 characterizes the set of allocations yielded by Bird's rule axiomatically, using efficiency, consistency and converse consistency. Section 4 presents a non-cooperative game, in which a strategy of a player consists of choosing how much to contribute to the cost of the links. Bird's tree allocations will turn out to be Nash equilibria of this game. Section 5 concludes.

Preliminaries and notations

We refer to any elementary textbook on graph theory for an understanding of graph theory, but recall some definitions to show the notational conventions. A graph $< V, E >$ consists of a set V of vertices and a set E of edges. An edge e incident with two vertices i and j is identified with $\{i, j\}$[3]. For a graph $< V, E >$ and a set $W \subseteq V$,

$$E(W) := \{e \in E \mid e \subseteq W\}$$

[3]Because we do not consider multigraphs : two vertices are connected by at most one edge.

is the set of edges linking two vertices in W. For a set $E' \subseteq E$,

$$V(E') := \{v \in V \mid \text{there exists an edge } e \in E' \text{ with } v \in e\}$$

is the set of vertices incident with E'.

The complete graph on a vertex set V is the graph $K_V =< V, E_V >$, where

$$E_V := \{\{v, w\} \mid v, w \in V \text{ and } v \neq w\}.$$

A *path* from i to j in a graph $< V, E >$ is a sequence $(i = i_0, i_1, \ldots, i_k = j)$ of vertices such that for all $l \leq k$, the edge $\{i_{l-1}, i_l\}$ lies in E. A *cycle* is a path of which the begin-point coincides with the end-point. Two vertices $i, j \in V$ are *connected* in a graph $< V, E >$ if there is a path from i to j in $< V, E >$. A subset W of V is *connected* in $< V, E >$ if every two vertices $i, j \in W$ are connected in the subgraph $< W, E(W) >$. A connected set W is a *component* of the graph $< V, E >$ if no superset of W is connected. A *connected graph* is a graph $< V, E >$ with V connected in $< V, E >$. A *tree* is a connected graph that contains no cycles. A *leaf* of a tree is a vertex that is incident to only one edge of the tree.

The cardinality of a set S will be denoted by $|S|$.

With many economic situations in which costs have to be divided one can associate a *cost game* (N, c) consisting of a finite set N of players, and a *characteristic function* $c : 2^N \to \mathbf{R}$, with $c(\emptyset) = 0$. Here $c(S)$ represents the minimal cost for coalition S if it secedes, i.e. if people of S cooperate and can not count upon help from people outside S.

The economic situations in the sequel involve a set N of users of a source $*$. For a coalition $S \subseteq N$, we denote $S \cup \{*\}$ by S^*. Furthermore, for a vector $x \in \mathbf{R}^N$ and a player $i \in N$, we denote x^{-i} the restriction of x to $N \setminus \{i\}$.

The *core* of a cost game (N, c), is defined by

$$\text{Core}(c) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N\}.$$

2 Mcst problems and Bird's tree allocation rule

Formally, a *minimum cost spanning tree (mcst) problem* $< N, *, w >$ consists of a finite group N of agents, each of whom wants to be connected to a common source, denoted by $*$. The non-negative cost of constructing a link $\{i, j\}$ between the vertices i and j in $N^* \equiv N \cup \{*\}$ is denoted by $w(i, j)$. Because of these costs, agents have an incentive to cooperate, and to construct a minimal cost graph that connects them all to the source. If a cycle appears in such a *minimum cost spanning graph*, at least one edge in this cycle can be eliminated, to yield another minimum cost spanning graph, with less cycles. Hence, there are minimum cost spanning graphs that contain no cycles at all, i.e. that are trees. This explains the name of the problem.

Prim (1957) and Dijkstra (1959) proposed the following algorithm to find a minimum cost spanning tree given an mcst problem.

Algorithm 2.1 (Prim and Dijkstra)

input : an mcst problem $\mathcal{T} \equiv < N, *, w >$
output : the edge set T of a minimum cost spanning tree

1. Choose a vertex $v \in N^*$ as *root*.

2. Initialize $T = \emptyset$.

3. Find a minimal cost edge $e \in E_{N^*} \setminus T$ incident to $\{v\} \cup N^*(T)$ such that joining e to T does not introduce a cycle.

4. Join e to T.

5. If not all vertices are connected to the root in the graph $< N^*, T >$, go back to stage 3.

Prim and Dijkstra then prove that any tree resulting from the algorithm is an mcst. Note that by varying between the possible edges in step 3, this algorithm can construct all minimum cost spanning trees of this mcst problem $\mathcal{T}$.

A problem related to such a minimization problem is how to allocate the cost of the edges of a minimum cost spanning tree among the agents (users of the source) in a reasonable way. Bird (1976) associated the following transferable utility mcst-game $(N, c^{\mathcal{T}})$ to an mcst problem $\mathcal{T}$. The players are the agents and the worth $c^{\mathcal{T}}(S)$ of a coalition S is the minimal cost of a tree on $S^* := S \cup \{*\}$. In formula,

$$c^{\mathcal{T}}(S) = \min\{\sum_{e \in T} w(e) \mid T \subseteq E_S \text{ and } < S^*, T > \text{ is a tree}\}$$

for all $S \subseteq N$. Bird also proposed a cost allocation rule for the mcst problem, which he calls the *tree allocation* rule, because it associates a cost allocation to every mcst of the mcst problem. Granot and Huberman (1981) proved that Bird's tree allocation rule yields extreme points of the core of the mcst game. Given an mcst problem $< N, *, w >$ and a mcst $< N^*, T >$ for the grand coalition, Bird's tree allocation β^T is constructed by assigning to a player $i \in N$ the cost of the first edge on the unique path in the tree $< N^*, T >$ from player i to the source $*$. In fact, this allocation is intimately linked with the Prim-Dijkstra algorithm : the tree $< N^*, T >$ and the allocation β^T can be constructed together by choosing the source as root and allocating the cost of the edge added at a certain stage to the person that this edge newly connects to the the source. More formally, the algorithm is the following.

Algorithm 2.2 (Bird's rule integrated into Prim and Dijkstra's algorithm)

input : an mcst problem $< N, *, w >$
output : an edge set T of an mcst and an allocation x (Bird's tree allocation β^T)

1. Choose the source $*$ as root.

2. Initialize $T = \emptyset$.

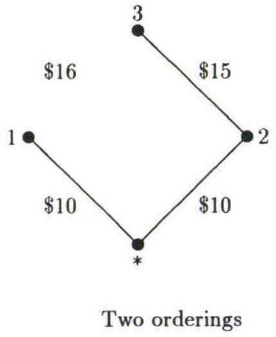
3. Find a minimal cost edge $e = \{i, j\} \in E_{N^*} \setminus T$ incident to $\{*\} \cup N^*(T)$ such that joining e to T does not introduce a cycle.

4. One of i and j, say j, was previously connected to the source and the other vertex, i, is a player that was not yet connected to the source. Assign the cost $x_i := w(e)$ to agent i.

5. Join e to T.

6. If not all vertices are connected to the root in the graph $< N^*, T >$, go back to stage 3.

As the set of all trees obtained by Prim and Dijkstra's algorithm is independent of the root that is chosen, this algorithm yields the same trees as Prim and Dijkstra's algorithm, and for each tree $< N^*, T >$, it yields an allocation that is Bird's tree allocation β^T associated to this tree. This is easy to see : in step 4, the edge e is precisely the first edge on the unique path from agent i to the source in the tree that will be constructed. If the mcst problem contains two or more edges with the same weight, there might be more than one mcst, and for a particular mcst $< N^*, T >$, it can happen that there is more than one order in which Prim-Dijkstra's algorithm can choose the edges in T. Obviously, the order does not change the edge that a player has to pay according to Bird's tree allocation rule, so Bird's tree allocation β^T is independent of the *order* in which the edges of the tree $< N^*, T >$ are chosen. It does, however, depend on which tree is constructed. See example 2.3.



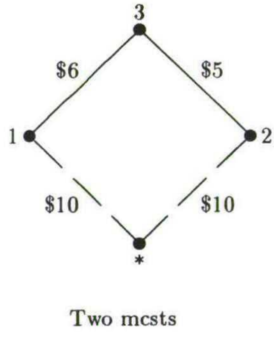


Figure 1: Edges that are not indicated cost $100.

Example 2.3 In the problem on the left of figure 1, whether Prim-Dijkstra's algorithm chooses the links in the unique mcst in the order $\{*, 1\}$, $\{*, 2\}$, $\{2, 3\}$ or $\{*, 2\}$, $\{*, 1\}$, $\{2, 3\}$, the link $\{*, 1\}$ is always paid by player 1 and the link $\{*, 2\}$ is always paid by player 2.

In the problem on the right of figure 1, only one of the two dashed links will be constructed. In case $\{*, 1\}$ is constructed, Bird's tree allocation is $(10, 5, 6)$ and if $\{*, 2\}$ is constructed, Bird's tree allocation is $(6, 10, 5)$.

3 An axiomatic characterization of Bird's tree allocation rule

In this section, we characterize the set of mcst and their associated Bird allocations axiomatically, using efficiency, consistency and converse consistency.

A *solution* of mcst problems is a function ψ assigning to every mcst problem $\mathcal{T} = < N, *, w >$, a set

$$\psi(\mathcal{T}) \subseteq \{((e^1, \dots, e^\tau), x) \mid < N^*, \{e^1, \dots, e^\tau\} > \text{ is connected and } \sum_{i \in N} x_i \geq \sum_{t=1}^{\tau} w(e^t)\}.$$

We mention a few properties of solutions of mcst problems.

Definition 3.1

NE A solution ψ is called *non-empty* if

$$\psi(\mathcal{T}) \neq \emptyset \qquad \text{for all mcst problems } \mathcal{T}.$$

Eff ψ is *efficient* if for all mcst problems $\mathcal{T}$, all $((e^1, \dots, e^\tau), x) \in \psi(\mathcal{T})$ are *efficient*, that is, for all $((e^1, \dots, e^\tau), x) \in \psi(\mathcal{T})$, $< N^*, \{e^1, \dots, e^\tau\} >$ is a minimal cost spanning tree and

$$\sum_{i \in N} x_i = \sum_{t=1}^{\tau} w(e^t).$$

The two properties of consistency and converse consistency use reduced mcst problems. Here, a reduced mcst problem is an mcst problem where some players have been eliminated. The idea is that solving reduced problems is easier than solving the original problem, and that the solution of the original problem should be related to the solution of the reduced problems. We only ask for this relation if one player that is a leaf is deleted, however. The idea is that a leaf is not needed by any other player to get connected to the source, so if a leaf player is missing, this should not affect the other players.

Definition 3.2 Given an mcst problem $\mathcal{T} \equiv < N, *, w >$ and a player $i \in N$, define the *reduced* mcst problem

$$\mathcal{T}^{-i} := < N \setminus \{i\}, *, w^{-i} >,$$

where w^{-i} is w restricted to $E_{N^* \setminus \{i\}}$.

Note that the reduced problem does not depend on any fixed solution. Note also that it is indeed an mcst problem. Using reduced mcst problems, we can define consistency and converse consistency as follows.

Definition 3.3

Cons A solution ψ of mcst problems is *consistent* if for every mcst problem $\mathcal{T}$, for every $((e^1,\ldots,e^\tau),x) \in \psi(\mathcal{T})$ and for every player i that is a leaf in the graph $< N^*, \{e^1,\ldots,e^\tau\} >$,

$$((e^1,\ldots,e^\tau)^{-i}, x^{-i}) \in \psi(\mathcal{T}^{-i}),$$

where $(e^1,\ldots,e^\tau)^{-i}$ is obtained from $(e^1,\ldots,e^\tau)$ by deleting the unique edge incident to i and x^{-i} is the vector obtained from x by deleting the coordinate of player i.

CoCons A solution ψ of mcst problems is *converse consistent* if for any mcst problem $\mathcal{T}$ and for any $((e^1,\ldots,e^\tau),x)$ efficient in $\mathcal{T}$, the following is satisfied : if

$$((e^1,\ldots,e^\tau)^{-i}, x^{-i}) \in \psi(\mathcal{T}^{-i})$$

for some player i that is a leaf of $< N^*, \{e^1,\ldots,e^\tau\} >$, then

$$((e^1,\ldots,e^\tau), x) \in \psi(\mathcal{T}).$$

The converse consistency property is motivated by the idea that if a possible efficient solution is excluded, the 'reduced' solution should also be excluded as solution in a reduced problem where a leaf has been deleted. It ensures that solutions that satisfy it are as large as possible without violating efficiency and consistency.

Definition 3.4 The Bird solution of an mcst problem $\mathcal{T}$ is the set

$$\beta(\mathcal{T}) := \{((e^1,\ldots,e^\tau), \beta^T(\mathcal{T})) \mid T = \{e^1,\ldots,e^\tau\} \text{ and } < N^*, T > \text{ is an mcst of } \mathcal{T}\}$$

of sequences of edges of minimum cost spanning trees and the corresponding Bird tree allocations.

Proposition 3.5 The Bird solution satisfies NE, Eff, Cons and CoCons.

Proof : Efficiency was proven by Bird and non-emptiness is evident. In order to prove Cons, assume $((e^1,\ldots,e^\tau),\beta^T) \in \beta(\mathcal{T})$ and let player i be a leaf in the tree $< N^*, T >$, where $T = \{e^1,\ldots,e^\tau\}$. Define e to be the first edge on the unique path in $< N^*, T >$ from i to the source. Then $(e^1,\ldots,e^\tau)^{-i}$ is obtained from $(e^1,\ldots,e^\tau)$ by deleting the edge e and is a sequence obtained by applying the Prim-Dijkstra algorithm to the reduced mcst problem $\mathcal{T}^{-i}$. Hence $x^{-i} = \beta^{T\setminus\{e\}}$, and

$$((e^1,\ldots,e^\tau)^{-i}, x^{-i}) \in \beta(\mathcal{T}^{-i}).$$

In order to prove that the Bird solution satisfies CoCons, assume that $((e^1,\ldots,e^\tau),x)$ is efficient in an mcst problem $\mathcal{T}$ and assume that player i is a leaf of $< N^*, \{e^1,\ldots,e^\tau\} >$ such that

$$((e^1,\ldots,e^\tau)^{-i}, x^{-i}) \in \beta(\mathcal{T}^{-i}). \tag{3.1}$$

Define e_i to be the unique edge incident to i in $\{e^1, \ldots, e^\tau\}$. Then $\{e^1, \ldots, e^\tau\} = \{e^1, \ldots, e^\tau\}^{-i} \cup \{e_i\}$ and $< N^* \setminus \{i\}, \{e^1, \ldots, e^\tau\}^{-i} >$ is an mcst for the reduced mcst problem $\mathcal{T}^{-i}$. Hence efficiency of $((e^1, \ldots, e^\tau), x)$ and equation 3.1 imply

$$\sum_{k \in N} x_k = \sum_{e \in \{e^1, \ldots, e^\tau\}} w(e) = \sum_{e \in \{e^1, \ldots, e^\tau\}^{-i}} w(e) + w(e_i) = \sum_{k \in N \setminus \{i\}} x_k^{-i} + w(e_i),$$

which implies that $x_i = w(e_i)$. So $((e^1, \ldots, e^\tau), x) \in \beta(\mathcal{T})$. □

Lemma 3.6 If a solution ϕ satisfies Eff and Cons and a solution ψ satisfies NE, Eff and CoCons, then $\phi(\mathcal{T}) \subseteq \psi(\mathcal{T})$ for all mcst problems $\mathcal{T}$.

Proof : We proceed by induction on the cardinality of N. Let $|N| = 1$ and denote by e the edge between the unique player and the source. By efficiency of both solutions and non-emptiness of ψ, we obtain $\phi(\mathcal{T}) \subseteq \{((e), w(e))\} = \psi(\mathcal{T})$. Take an mcst problem $\mathcal{T}$ with $k > 1$ players, and suppose that for all mcst problems $\mathcal{T}'$ with less than k players, $\phi(\mathcal{T}') \subseteq \psi(\mathcal{T}')$. Take $((e^1, \ldots, e^\tau), x) \in \phi(\mathcal{T})$ and choose a leaf $i \neq *$ of the tree T induced by $(e^1, \ldots, e^\tau)$. Then by consistency of ϕ, $((e^1, \ldots, e^\tau)^{-i}, x^{-i}) \in \phi(\mathcal{T}^{-i}) \subseteq \psi(\mathcal{T}^{-i})$. Now $((e^1, \ldots, e^\tau), x)$ is efficient, hence $((e^1, \ldots, e^\tau), x) \in \psi(\mathcal{T})$ by converse consistency of ψ. □

Theorem 3.7 The unique solution that satisfies NE, Eff, Cons, and CoCons is the Bird solution.

Proof : The Bird solution has the properties, and if another solution has the properties, by lemma 3.6, it coincides with the Bird solution. □

The properties used to characterize the Bird solution are logically independent. We show this by giving examples of solutions that satisfy three of the four properties.

Example 3.8 If we leave out the non-emptiness property, the empty solution that assigns the empty set to every mcst problem satisfies Eff, Cons and CoCons.

Example 3.9 If we leave out the efficiency property, the solution that assigns $((e)_{e \in E_{N^*}}, (a, \ldots, a))$ to every mcst problem, satisfies the other three properties. Here $(e)_{e \in E_{N^*}}$ denotes the sequence of all edges of the complete graph on N^* ordered by non-decreasing magnitude, and $a = sum_{e \in E_{N^*}} w(e)$. Notice that there are no leaves in the complete graph, except if there is only one player, so the consistency property is satisfied vacuously.

Example 3.10 If we leave out consistency, the solution that assigns to an mcst problem $< N, *, w >$ the set of all efficient outcomes

$$\{((e^1, \ldots, e^\tau), x) \mid < N^*, \{e^1, \ldots, e^\tau\} > \text{ is an mcst, } x \in \mathbf{R}^N \text{ and } x(N) = \sum_{t=1}^{\tau} w(e^t)\}$$

satisfies the three other properties.

For the last example, we assume there is a total ordering $<$ on the set of all possible players. This is a reasonable assumption : usually, names of players are finite strings in some finite alphabet, which can be alphabetically ordered. Define the lexicographical order on elements of a solution to an mcst problem $< N, *, w >$ by $((e^1, \ldots, e^\tau), x) \prec_L ((\tilde{e}^1, \ldots, \tilde{e}^{\tilde{\tau}}), y)$ if there exists a $k \in N$ such that $x_i = y_i$ for $i < k$ and $x_k < y_k$.

Example 3.11 If we leave out converse consistency, the solution that assigns to every mcst problem the set of lexicographically smallest elements of the Bird solution satisfies the three other properties, but does not coincide with the Bird solution on all mcst problems, so it does not satisfy converse consistency.

4 Sustaining the Bird tree allocations by Nash equilibria

In the previous sections we studied mcst problems by means of cooperative games. In this section, we model the problems by strategic games, in which an action of a player consists of a specification of the edges to which this player will contribute, and which amounts he will contribute.

Definition 4.1 To an minimum cost spanning tree problem $< N, *, w >$, we associate the strategic game $< N, (A^i)_{i \in N}, (u_i)_{i \in N} >$ in normal form with player set N, and in which an action $a^i = (a^i_j)_{j \in N^* \setminus \{i\}} \in A^i \equiv \mathbf{R}_+^{N^* \setminus \{i\}}$ of a player i specifies for each other vertex j (j can be a player or the source), the non-negative amount a^i_j that player i is willing to contribute to the cost of the edge $\{i, j\}$. The utility that player i derives from a strategy profile $a = (a^i)_{i \in N}$ is determined in the following way. We assume that players dislike making contributions, but they absolutely have to be connected to the source. So the utility function is linear in the contributions, and a big penalty is subtracted if the player is not connected to the source in the graph resulting from the contributions of all players. More precisely, for a strategy profile a, the set C_a of edges that have been completely paid for and that will be constructed is defined as

$$C_a := \{\{i, j\} \in E_{N^*} \mid a^i_j + a^j_i \geq w(e)\}$$

and the utility of player i is defined as

$$u_i(a^1, \ldots, a^n) := \begin{cases} -\sum\limits_{j \in N^* \setminus \{i\}} a^i_j & \text{if } i \text{ is connected to the source in } < N^*, C_a > \\ -\sum\limits_{j \in N^* \setminus \{i\}} a^i_j - P & \text{otherwise} \end{cases}$$

where P is a large number ($P > \sum\limits_{e \in E_{N^*}} w(e)$).

We proceed to establish a relationship between the Bird solution presented in section 2 and Nash equilibria of the above strategic mcst game.

Theorem 4.2 Each element $((e^1,\dots,e^\tau),x)$ of the Bird solution of an mcst problem corresponds to a Nash equilibrium of the associated strategic mcst game, in which the strategy of a player i is to construct the first edge on the unique path from i to the source in the tree $< N^*, \{e^1,\dots,e^\tau\} >$ and his payoff equals $-x_i$.

Proof : Let $\mathcal{T} = < N, *, w >$ be an mcst problem and let $((e^1,\dots,e^\tau),x)$ be an element of the Bird solution $\beta(\mathcal{T})$. The corresponding strategy a^i for a player i is to contribute only to the first edge e_i that lies on the unique path from i to the source in the tree $< N^*, \{e^1,\dots,e^\tau\} >$ and to pay the cost $w(e_i)$ of this edge completely. If every player plays this strategy, the resulting set C_a of constructed edges is precisely $\{e^1,\dots,e^\tau\}$, which implies that all players are connected to the source. So the payoff to player i equals $-w(e_i)$. Hence $u_i(a) = -w(e_i) = -x_i$.

To prove that a is a Nash equilibrium, suppose that a player i deviates from a. It is clear that contributing more to the edge e_i does not improve i's payoff. If i contributes less than $w(e_i)$ to the edge e_i, then e_i is not paid completely, so it will not be constructed. Now player i wants to avoid the penalty, which is larger than $w(e_i)$, so i has to pay at least one other edge e' that connects the component of i in the graph $< N^*, C_a \setminus \{e_i\} >$ to the component of the source. Because $< N^*, \{e^1,\dots,e^\tau\} >$ is an mcst, such an edge e' has to be at least as costly as the edge e_i. Hence i is not better off. □

If the costs of all edges are positive, it can be proved that in all Nash equilibria of the strategic game a spanning tree is formed, although it does not have to be a mcst.

Theorem 4.3 If in an mcst problem the costs of all edges are positive, each Nash equilibrium a of the strategic mcst game specifies a spanning tree $< N^*, C_a >$ for the mcst problem, and the payoff vector is $-\beta^{C_a}$. Here, for any spanning tree $< N^*, T >$ we again denote β^T the allocation assigning to every player the cost of the first edge on the unique path from the player to the source in this tree.

Proof : Let $< N, *, w >$ be an mcst problem in which the costs of all edges is positive and let $< N, (A^i)_{i\in N}, (u_i)_{i\in N} >$ be the associated strategic mcst game. Let $a = (a^1,\dots,a^n)$ be a Nash equilibrium and consider the set C_a of edges that have been completely paid. If a player i is not connected to the source in the graph $< N^*, C_a >$, then by deviating and using the strategy $\hat{a}$ in which he pays precisely the link $\{i,*\}$, he can improve his payoff. So in a Nash equilibrium, every player is connected to the root. Furthermore, if a cycle were present in the graph $< N^*, C_a >$, there is a player i that contributes a positive amount to an edge of this cycle. Then i can improve his payoff by not contributing to this edge. Hence $< N^*, C_a >$ is a tree.

It is clear that no edge of the constructed tree will be 'overpaid' and that no other edge will be contributed to. Furthermore, every player i contributes only to edges that lie on the path in the tree from i to the source. If this were not true, some player could reduce his contribution to an edge that does not lie on the path from himself to the source, without incurring the penalty. Now there is only one edge in the path from i to the source in the tree that i can contribute to, and that is the edge $e = \{i,j\}$ in the tree, that is the first edge on the unique path in the tree from i to the source. By an induction argument one sees that no other player contributes anything to this edge, which implies that i pays $w(e)$ alone. So $a^i_j = w(e)$ and $a^i_k = 0$ for all other $k \in N^*$, and the payoff is $u_i(a) = -w(e) = \beta^{C_a}$. □

Not all Nash equilibria correspond to mcsts, as is shown by the next example.

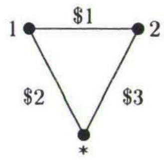


Figure 2: A problem with a Nash equilibrium that does not correspond to an mcst.

Example 4.4 Consider the problem drawn in figure 2. The strategy pair where player 1 pays the edge $\{1,2\}$ and player 2 pays $\{2,*\}$ is a Nash equilibrium in the associated strategic game, but the unique mcst uses the edge $\{1,*\}$ instead of $\{2,*\}$.

We call a Nash equilibrium *total payoff maximizing* if the sum of the payoffs to the players in this equilibrium is maximal among all Nash equilibria.

Theorem 4.5 In an mcst problem with positive weights, the total payoff maximizing Nash equilibria correspond to mcsts.

Proof : We know that a Nash equilibrium corresponds to a tree, and that the payoffs of the players correspond to costs of edges in the tree. If the total of the payoffs is maximal, the cost of the tree is minimal, hence the tree is a mcst. □

Remark Total payoff maximizing Nash equilibria are strong Nash equilibria (private communication by Gert-Jan Otten). In a strong equilibrium, deviations by coalitions of players do not strictly improve the payoffs of all players in the coalition. Hence, theorem 4.2 can be replaced by a stronger theorem, viz.,

Theorem 4.6 Each element $((e^1, \ldots, e^\tau), x)$ of the Bird solution of an mcst problem corresponds to a strong Nash equilibrium of the associated strategic mcst game, in which the strategy of a player i is to construct the first edge on the unique path from i to the source in the tree $< N^*, \{e^1, \ldots, e^\tau\} >$ and his payoff equals $-x_i$.

However, the non-total-payoff-maximizing equilibrium in example 4.4 is a strong Nash equilibrium. Hence, the set of all strong equilibria does not coincide with the set of all total payoff maximizing equilibria in all strategic mcst games.

5 Conclusion

In this paper, we have reconsidered Bird's tree allocations for mcst problems, and have studied them from two points of view. First we gave an axiomatic characterization which permits us to determine whether the allocation rule is applicable. Second we introduced a non-cooperative game, of which the total payoff maximizing Nash equilibria coincide with Bird's tree allocations.

Instead of using Bird's tree allocation rule, one could use elements of the irreducible core of a mcst problem to evaluate it. The irreducible core was also introduced by Bird (1976) and studied by Granot and Huberman (1982) and Aarts and Driessen (1993). We axiomatically characterize the irreducible core of a generalization of mcst problems in Feltkamp, Muto and Tijs (1994a), and also prove that it can be obtained as the set of all allocations that are associated to Kruskal's (1956) algorithm to construct minimum cost spanning trees.

Knowing that there are several algorithms to construct minimum cost spanning trees, one can consider associating other allocations to them. This is done in Feltkamp, Tijs and Muto (1994a) and (1994b) for Kruskal's algorithm, and an older algorithm described in Borůvka (1926).

Acknowledgements : Inspiring discussions with Peter Borm, Michael Maschler, Gert-Jan Otten, Jos Potters, Hans Reijnierse and Oscar Volij about the subject of the paper are gratefully acknowledged.

References

Aarts H (1992) Marginal allocations in the core of minimal chain games, *working paper no. 1103, Dept. of Applied Math, University of Twente, Enschede, The Netherlands.*

Aarts H and Driessen Th (1993) The irreducible core of a minimum cost spanning tree game, *Zeitschrift für Operations Research* 38 : 163–174.

Bird CG (1976) On cost allocation for a spanning tree : A game theoretic approach. *Networks* 6 : 335–350.

Borůvka O (1926a) O jistém problému minimálním. *Práce Mor. Přírodověd. Spol. v Brně (Acta Societ. Scient Natur. Moravicae)* 3 : 37–58.

Borůvka O (1926b) Příspěvek k řešení otázky ekonomické stavby elektrovodních sítí. *Elektrotechnický obzor* 15 : 153–154.

Claus A and Kleitman DJ (1973) Cost allocation for a spanning tree. *Networks* 3 : 289–304.

Dijkstra EW (1959) A note on two problems in connection with graphs, *Numerische Mathematik* 1 : 269–271.

Feltkamp V, Muto S, and Tijs S (1994a) On the irreducible core of minimum cost spanning extension games, *mimeo*, Tilburg University.

Feltkamp V, Muto S, and Tijs S (1994b) Core allocations of minimum cost spanning extension games associated to Kruskal's and Borůvka's algorithms. *mimeo*, Tilburg University.

Graham RL and Hell P (1985) On the history of the minimum spanning tree problem. *Annals of the History of Computing* 7 : 43–57.

Granot D and Huberman G (1981) On minimum cost spanning tree games, *Mathematical programming* 21 : 1–18.

Granot D and Huberman G (1982) The relationship between convex games and minimal cost spanning tree games : A case for permutationally convex games, *SIAM J. Alg. and Discr. Meth.* 3 : 288–292.

Granot D and Huberman G (1984) On the core and nucleolus of minimum cost spanning tree games, *Mathematical programming* 29 : 323–347.

Kuipers (1993) On the core of information graph games. *Int. J. of Game Th.* 21 : 339–350.

Kruskal (1956) On the shortest spanning subtree of a graph and the traveling salesman problem, *Proc. American Math. Soc.* 7 : 48–50.

Prim RC (1957) Shortest connection networks and some generalizations, *Bell Systems Techn. J.* 36 : 1389–1401.

Discussion Paper Series, CentER, Tilburg University, The Netherlands:

(For previous papers please consult previous discussion papers.)

No.	Author(s)	Title
9324	H. Huizinga	The Financing and Taxation of U.S. Direct Investment Abroad
9325	S.C.W. Eijffinger and E. Schaling	Central Bank Independence: Theory and Evidence
9326	T.C. To	Infant Industry Protection with Learning-by-Doing
9327	J.P.J.F. Scheepens	Bankruptcy Litigation and Optimal Debt Contracts
9328	T.C. To	Tariffs, Rent Extraction and Manipulation of Competition
9329	F. de Jong, T. Nijman and A. Röell	A Comparison of the Cost of Trading French Shares on the Paris Bourse and on SEAQ International
9330	H. Huizinga	The Welfare Effects of Individual Retirement Accounts
9331	H. Huizinga	Time Preference and International Tax Competition
9332	V. Feltkamp, A. Koster, A. van den Nouweland, P. Borm and S. Tijs	Linear Production with Transport of Products, Resources and Technology
9333	B. Lauterbach and U. Ben-Zion	Panic Behavior and the Performance of Circuit Breakers: Empirical Evidence
9334	B. Melenberg and A. van Soest	Semi-parametric Estimation of the Sample Selection Model
9335	A.L. Bovenberg and F. van der Ploeg	Green Policies and Public Finance in a Small Open Economy
9336	E. Schaling	On the Economic Independence of the Central Bank and the Persistence of Inflation
9337	G.-J. Otten	Characterizations of a Game Theoretical Cost Allocation Method
9338	M. Gradstein	Provision of Public Goods With Incomplete Information: Decentralization vs. Central Planning
9339	W. Güth and H. Kliemt	Competition or Co-operation
9340	T.C. To	Export Subsidies and Oligopoly with Switching Costs
9341	A. Demirgüç-Kunt and H. Huizinga	Barriers to Portfolio Investments in Emerging Stock Markets
9342	G.J. Almekinders	Theories on the Scope for Foreign Exchange Market Intervention

No.	Author(s)	Title
9343	E.R. van Dam and W.H. Haemers	Eigenvalues and the Diameter of Graphs
9344	H. Carlsson and S. Dasgupta	Noise-Proof Equilibria in Signaling Games
9345	F. van der Ploeg and A.L. Bovenberg	Environmental Policy, Public Goods and the Marginal Cost of Public Funds
9346	J.P.C. Blanc and R.D. van der Mei	The Power-series Algorithm Applied to Polling Systems with a Dormant Server
9347	J.P.C. Blanc	Performance Analysis and Optimization with the Power-series Algorithm
9348	R.M.W.J. Beetsma and F. van der Ploeg	Intramarginal Interventions, Bands and the Pattern of EMS Exchange Rate Distributions
9349	A. Simonovits	Intercohort Heterogeneity and Optimal Social Insurance Systems
9350	R.C. Douven and J.C. Engwerda	Is There Room for Convergence in the E.C.?
9351	F. Vella and M. Verbeek	Estimating and Interpreting Models with Endogenous Treatment Effects: The Relationship Between Competing Estimators of the Union Impact on Wages
9352	C. Meghir and G. Weber	Intertemporal Non-separability or Borrowing Restrictions? A Disaggregate Analysis Using the US CEX Panel
9353	V. Feltkamp	Alternative Axiomatic Characterizations of the Shapley and Banzhaf Values
9354	R.J. de Groof and M.A. van Tuijl	Aspects of Goods Market Integration. A Two-Country-Two -Sector Analysis
9355	Z. Yang	A Simplicial Algorithm for Computing Robust Stationary Points of a Continuous Function on the Unit Simplex
9356	E. van Damme and S. Hurkens	Commitment Robust Equilibria and Endogenous Timing
9357	W. Güth and B. Peleg	On Ring Formation In Auctions
9358	V. Bhaskar	Neutral Stability In Asymmetric Evolutionary Games
9359	F. Vella and M. Verbeek	Estimating and Testing Simultaneous Equation Panel Data Models with Censored Endogenous Variables
9360	W.B. van den Hout and J.P.C. Blanc	The Power-Series Algorithm Extended to the *BMAP/PH/*1 Queue
9361	R. Heuts and J. de Klein	An (s,q) Inventory Model with Stochastic and Interrelated Lead Times

No.	Author(s)	Title
9362	K.-E. Wärneryd	A Closer Look at Economic Psychology
9363	P.J.-J. Herings	On the Connectedness of the Set of Constrained Equilibria
9364	P.J.-J. Herings	A Note on "Macroeconomic Policy in a Two-Party System as a Repeated Game"
9365	F. van der Ploeg and A. L. Bovenberg	Direct Crowding Out, Optimal Taxation and Pollution Abatement
9366	M. Pradhan	Sector Participation in Labour Supply Models: Preferences or Rationing?
9367	H.G. Bloemen and A. Kapteyn	The Estimation of Utility Consistent Labor Supply Models by Means of Simulated Scores
9368	M.R. Baye, D. Kovenock and C.G. de Vries	The Solution to the Tullock Rent-Seeking Game When $R > 2$: Mixed-Strategy Equilibria and Mean Dissipation Rates
9369	T. van de Klundert and S. Smulders	The Welfare Consequences of Different Regimes of Oligopolistic Competition in a Growing Economy with Firm-Specific Knowledge
9370	G. van der Laan and D. Talman	Intersection Theorems on the Simplotope
9371	S. Muto	Alternating-Move Preplays and vN - M Stable Sets in Two Person Strategic Form Games
9372	S. Muto	Voters' Power in Indirect Voting Systems with Political Parties: the Square Root Effect
9373	S. Smulders and R. Gradus	Pollution Abatement and Long-term Growth
9374	C. Fernandez, J. Osiewalski and M.F.J. Steel	Marginal Equivalence in ν-Spherical Models
9375	E. van Damme	Evolutionary Game Theory
9376	P.M. Kort	Pollution Control and the Dynamics of the Firm: the Effects of Market Based Instruments on Optimal Firm Investments
9377	A. L. Bovenberg and F. van der Ploeg	Optimal Taxation, Public Goods and Environmental Policy with Involuntary Unemployment
9378	F. Thuijsman, B. Peleg, M. Amitai & A. Shmida	Automata, Matching and Foraging Behavior of Bees
9379	A. Lejour and H. Verbon	Capital Mobility and Social Insurance in an Integrated Market

No.	Author(s)	Title
9380	C. Fernandez, J. Osiewalski and M. Steel	The Continuous Multivariate Location-Scale Model Revisited: A Tale of Robustness
9381	F. de Jong	Specification, Solution and Estimation of a Discrete Time Target Zone Model of EMS Exchange Rates
9401	J.P.C. Kleijnen and R.Y. Rubinstein	Monte Carlo Sampling and Variance Reduction Techniques
9402	F.C. Drost and B.J.M. Werker	Closing the Garch Gap: Continuous Time Garch Modeling
9403	A. Kapteyn	The Measurement of Household Cost Functions: Revealed Preference Versus Subjective Measures
9404	H.G. Bloemen	Job Search, Search Intensity and Labour Market Transitions: An Empirical Exercise
9405	P.W.J. De Bijl	Moral Hazard and Noisy Information Disclosure
9406	A. De Waegenaere	Redistribution of Risk Through Incomplete Markets with Trading Constraints
9407	A. van den Nouweland, P. Borm, W. van Golstein Brouwers, R. Groot Bruinderink, and S. Tijs	A Game Theoretic Approach to Problems in Telecommunication
9408	A.L. Bovenberg and F. van der Ploeg	Consequences of Environmental Tax Reform for Involuntary Unemployment and Welfare
9409	P. Smit	Arnoldi Type Methods for Eigenvalue Calculation: Theory and Experiments
9410	J. Eichberger and D. Kelsey	Non-additive Beliefs and Game Theory
9411	N. Dagan, R. Serrano and O. Volij	A Non-cooperative View of Consistent Bankruptcy Rules
9412	H. Bester and E. Petrakis	Coupons and Oligopolistic Price Discrimination
9413	G. Koop, J. Osiewalski and M.F.J. Steel	Bayesian Efficiency Analysis with a Flexible Form: The AIM Cost Function
9414	C. Kilby	World Bank-Borrower Relations and Project Supervision
9415	H. Bester	A Bargaining Model of Financial Intermediation
9416	J.J.G. Lemmen and S.C.W. Eijffinger	The Price Approach to Financial Integration: Decomposing European Money Market Interest Rate Differentials

No.	Author(s)	Title
9417	J. de la Horra and C. Fernandez	Sensitivity to Prior Independence via Farlie-Gumbel-Morgenstern Model
9418	D. Talman and Z. Yang	A Simplicial Algorithm for Computing Proper Nash Equilibria of Finite Games
9419	H.J. Bierens	Nonparametric Cointegration Tests
9420	G. van der Laan, D. Talman and Z. Yang	Intersection Theorems on Polytopes
9421	R. van den Brink and R.P. Gilles	Ranking the Nodes in Directed and Weighted Directed Graphs
9422	A. van Soest	Youth Minimum Wage Rates: The Dutch Experience
9423	N. Dagan and O. Volij	Bilateral Comparisons and Consistent Fair Division Rules in the Context of Bankruptcy Problems
9424	R. van den Brink and P. Borm	Digraph Competitions and Cooperative Games
9425	P.H.M. Ruys and R.P. Gilles	The Interdependence between Production and Allocation
9426	T. Callan and A. van Soest	Family Labour Supply and Taxes in Ireland
9427	R.M.W.J. Beetsma and F. van der Ploeg	Macroeconomic Stabilisation and Intervention Policy under an Exchange Rate Band
9428	J.P.C. Kleijnen and W. van Groenendaal	Two-stage versus Sequential Sample-size Determination in Regression Analysis of Simulation Experiments
9429	M. Pradhan and A. van Soest	Household Labour Supply in Urban Areas of a Developing Country
9430	P.J.J. Herings	Endogenously Determined Price Rigidities
9431	H.A. Keuzenkamp and J.R. Magnus	On Tests and Significance in Econometrics
9432	C. Dang, D. Talman and Z. Wang	A Homotopy Approach to the Computation of Economic Equilibria on the Unit Simplex
9433	R. van den Brink	An Axiomatization of the Disjunctive Permission Value for Games with a Permission Structure
9434	C. Veld	Warrant Pricing: A Review of Empirical Research
9435	V. Feltkamp, S. Tijs and S. Muto	Bird's Tree Allocations Revisited

PO BOX 90153 5000 LE TILBURG THE NETHERLANDS

Bibliotheek K. U. Brabant

17 000 01158452 2